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Similarity solutions in fragmentation kinetics

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Abstract. For different homogeneous fragmentation models we derive a common integro-differential equation. We assume that the fragmentation rate is homogeneous of order γ in both arguments. A reduction of the integro-differential equation to a partial differential equation delivers different types of similarity solutions by applying Lie's similarity method. Calculation of the adjoint algebra performs a classification of the non-trivial solutions.

1. Introduction

In several fields of physics (polymer science, mineralogy, combustion theory, neutron transport theory) the fragmentation process is of considerable interest. The early work on this theme was done by Kuhn (1930), Mark and Simha (1940), and Montroll and Simha (1940) in polymer science. These authors determined the distribution function by statistical methods considering 'random scission' processes where all bonds break with equal probability (Montroll and Simha 1940). Later on Saito (1958) and Jellinek and White (1951) gave continuous and discrete models, respectively, of a fragmentation process. The continuous model introduced by Saito (1958) has a wide application in physics; for references see Peterson (1986).

Much of the theoretical work is based on the description of fragmentation by a system of linear rate equations in the discrete form which are suitable for numerical analysis. For an analytical treatment of fragmentation the continuous models are more appropriate. The continuous models are typically represented by a linear integro-differential balance equation. In a generalized nonlinear model Cheng and Redner (1988) proposed a fragmentation process caused by repeated collisions between clusters. Amemiya (1962) introduced on the other hand an inhomogeneity by having bonds of different breaking probability dispersed throughout the system. Dependence of the scission rate on the size of the chain was considered by Basedow *et al* (1978) and Ballauf and Wolf (1981) in theoretical and numerical works.

In our considerations we are interested in models where the breakup rate depends upon size. In such models the cutting is non-random in the sense that bonds on different chains break with different probability. We examine continuous models with homogeneous breakup rates in the kinetic equation. We find that the asymptotic form of the size distribution at large segment size is strongly determined by the homogeneity index of the breakup kernel. The continuous model is also somewhat more interesting analytically, since the determination of the solutions is for some models quite obvious. Special solutions for fragmentation of the continuous model have been discussed by

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McGrady and Ziff (1987), and Corngold and Williams (1989). In a work on similarity solutions Peterson (1986) applied the fragmentation model to particle fragmentation.

Starting out from the kinetic equation of Ziff and McGrady (1985) with homogeneous forms of the breakup kernel we will determine general similarity solutions based on Lie's method. When particle breakup takes place independently and homogeneously, the evolution of the number of chains $n(x, t)$ of continuous length x at time t can be described by

$$\frac{\partial n(x,t)}{\partial t} = -n(x,t)\int_0^x F(y, x-y)\,dy + 2\int_x^\infty n(y,t)F(x, y-x)\,dy \qquad (1)$$

where $F(x, y)$ gives the rate that a segment breaks up into two parts of length x and y. For details of the model see Ziff and McGrady (1985). Since equation (1) is linear it can be solved in principle for any breakup function. Several non-homogeneous breakup functions are given by Peterson (1986). Our considerations here will only take into account homogeneous breakup kernels.

The paper is organized as follows. First we will demonstrate that certain models with homogeneous breakup kernel can be cast into a common equation. In section 2 we give a similarity solution for the generalized model. With the similarity solution on hand we can demonstrate that for a subclass of homogeneous kernels $F(x, y)$ a scaling behaviour is possible. To characterize the asymptotic behaviour of our solutions for large t and small x we consider the self-similar properties of these results. We demonstrate that a similarity solution for homogeneous kernels $F(x, y)$ can be given by $u(x, t) = s^{1/\lambda}(t)v(\zeta)$, where the similarity variable ζ is given by $\zeta = tx^{\gamma+1}$, and s is a function of time. This type of solution corresponds to an infinitesimal scaling invariance under which the corresponding partial differential equation (PDE) of equation (1) is invariant for homogeneous kernels. Another type of similarity solution is given by $u(x, t) = \exp(\frac{1}{2}act^2)v(\zeta)$ with $\zeta = x^{\gamma+1} - ct$ which is a moving wave solution for $\gamma = 0$.

2. Breakup kernels of $F(\lambda x, \lambda y) = \lambda^\gamma F(x, y)$

Let us first consider systems possessing $F(x, y) = (x+y)^\alpha$ as kernel. Here the rate of breakup is proportional to the length of the chain to the power α. Setting $\alpha = 0$, the equi-reactivity model of Montroll and Simha (1940) is recovered. In a further examination, we discuss a model for which the breakup depends upon the size of the second object breaking up $(x+y)$ and upon the individual size of the first piece. The breakup kernel for such a process is given by $F(x, y) = x^\nu(x+y)^{\beta-\nu}$ considered by McGrady and Ziff (1987). A special case of this model is obtained if we set $\beta = 2\nu$. To demonstrate the homogeneity of the kernels $F(x, y)$ we have to consider the equations $F(\lambda x, \lambda y) = \lambda^\gamma F(x, y)$. Performing the simple calculation one obtains

(i) $F(x, y) = (x+y)^\alpha$: $F(\lambda x, \lambda y) = \lambda^\alpha F(x, y)$

(ii) $F(x, y) = x^\nu(y+x)^{\beta-\nu}$: $F(\lambda x, \lambda y) = \lambda^\beta F(x, y)$

(iii) $F(x, y) = x^\nu(y+x)^\nu$: $F(\lambda x, \lambda y) = \lambda^{2\nu} F(x, y)$.

The exponents α, β and ν determine the degree of homogeneity of these models. Determining the kinetic equation for each model (i)–(iii) we can write

$$n_t(x, t) = -x^{\alpha+1}n(x, t) + 2\int_x^\infty y^\alpha n(y, t)\,dy \qquad (2)$$

for model (i), and

$$n_t(x,t) = -\frac{1}{\nu+1}x^{\beta+1}n(x,t) + 2x^{\nu}\int_x^{\infty} y^{\beta-\nu}n(y,t)\,dy \tag{3}$$

for model (ii) and (iii), which is closely related to the model of McGrady and Ziff (1987) with an asymmetric kernel. The physical conditions under which (3) is valid are also given in the cited paper. Here we will give a straightforward procedure to solve this equation. If one multiplies equation (3) with a factor $x^{-\nu}$ and substitutes $g(x,t) = x^{-\nu}n(x,t)$ in (3) it is possible to reformulate (3) to

$$g_t(x,t) = -\frac{1}{\nu+1}x^{\beta+1}g(x,t) + 2\int_x^{\infty} y^{\beta}g(y,t)\,dy. \tag{4}$$

Comparing the equations (2) and (4) we conclude that it is crucial to solve an equation like

$$u_t(x,t) = -dx^{\gamma+1}u(x,t) + 2\int_x^{\infty} y^{\gamma}u(y,t)\,dy \tag{5}$$

where

$$d = \begin{cases} 1 & \text{for (i)} \\ 1/(\nu+1) & \text{for (ii) and (iii)} \end{cases} \tag{6}$$

and

$$\gamma = \begin{cases} \alpha & \text{for (i)} \\ \beta & \text{for (ii)} \\ 2\nu & \text{for (iii).} \end{cases} \tag{7}$$

We have to assume that at least $u(x\to\infty, t)$ goes stronger to 0 than $x^{-\gamma-1}$ to be on the safe side respecting the convergence of the integral in (5). The general solutions for each model are then given by

$$n(x,t) = u(x,t) \qquad \text{for (i)}$$

$$n(x,t) = x^{\nu}u(x,t) \qquad \text{for (ii) and (iii)}$$

respectively. The corresponding PDE to (5) is obtained by differentiating with respect to x

$$u_{tx}(x,t) = -x^{\gamma}(d(\gamma+1)+2)u(x,t) - dx^{\gamma+1}u_x(x,t) \tag{8}$$

where d and γ are given by (6) and (7). If we set $d=1$ and $\gamma=0$ we obtain the equi-reactivity model of Montroll and Simha (1940) in the form

$$u_{tx} = -xu_x - 3u. \tag{9}$$

In the following we will apply the similarity method to (8) in order to obtain solutions for the different models.

3. Similarity-solution of the kinetic equation

The common kinetic equation for homogeneous scaling kernels is given by

$$u_t(x,t) = -dx^{\gamma+1}u(x,t) + 2\int_x^{\infty} y^{\gamma}u(y,t)\,dy. \tag{10}$$

Equation (10) is equivalent to the partial differential equation (PDE)

$$u_{xt} = Bx^{\gamma}u(x, t) + Ax^{\gamma+1}u_x \tag{11}$$

where $B = -\{d(\gamma+1)+2\}$, and $A = -d$. To consider the symmetries of the linear PDE (11) we have to examine a one-parameter group G of transformations

$$x^* = f(x, u; \varepsilon) \qquad u^* = \phi(x, u; \varepsilon) \tag{12}$$

in the space R^{n+m} with variables $x = (x^1, \ldots, x^n)$ and $u = (u^1, \ldots, u^m)$. In our case we have $n = 2$ and $m = 1$. The identity of transformation (12) is given by $x = f|_{\varepsilon=0}$ and $u = \phi|_{\varepsilon=0}$. Now take the additional variables $u_1 = \{u_i^{\alpha} \mid \alpha = 1, \ldots, m; i = 1, \ldots, n\}$ and subject them to the transformations

$$u_i^{*\alpha} = \psi_i^{\alpha}(x, u, u_1; \varepsilon) \tag{13}$$

with $\psi_i^{\alpha}|_{\varepsilon=0} = u_i^{\alpha}$. We require that (13) and transformations of the derivatives $\partial u^{\alpha}/\partial x^i$ under the change of variables (12) be compatible with the equalities $u_i^{\alpha} = \partial u^{\alpha}/\partial x^i$ for any function $u^{\alpha} = u^{\alpha}(x)$. This condition uniquely defines the transformation (13) for any Lie group G, and the result is a one-parameter group G_1 of transformations (12), (13) acting in the space R^{n+m+nm} with variables (x, u, u_1). The mappings (12) are called point transformations whereas (13) are the prolongations of these point transformation. G_1 is *the first prolongation of the group G*.

Let

$$v = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^{\alpha}(x, u)\frac{\partial}{\partial u^{\alpha}} \tag{14}$$

be the infinitesimal operator or vector field of the group G, where

$$\xi = \left.\frac{\partial f}{\partial \varepsilon}\right|_{\varepsilon=0} \qquad \text{and} \qquad \eta = \left.\frac{\partial \phi}{\partial \varepsilon}\right|_{\varepsilon=0} \tag{15}$$

are the infinitesimal elements of transformation (12). The infinitesimal operator of the *first prolongation* G_1 *is*

$$\mathbf{pr}^{(1)}(v) = v + \eta_i^{\alpha}\frac{\partial}{\partial u_i^{\alpha}} \tag{16}$$

where $\eta_i^{\alpha} = \partial\psi_i^{\alpha}/\partial\varepsilon|_{\varepsilon=0}$ are prolongation elements of u_1. We assume summation over repeated indices. The first and all higher prolongation elements η_i^{α}, η_{ij}^{α} can be determined recursively from the infinitesimals ξ and η by

$$\eta_i^{\alpha} = D_i(\eta^{\alpha}) - u_j^{\alpha}D_i(\xi^j). \tag{17}$$

Here $D_i = \partial/\partial x^i + u_i^{\alpha}\partial/\partial u^{\alpha}$ is the operator of total differentiation with respect to the variable x^i. The second prolongation elements η_{ij}^{α} follow from

$$\eta_{ij}^{\alpha} = D_j(\eta_i^{\alpha}) - u_{ik}^{\alpha}D_j(\xi^k). \tag{18}$$

Since our problem contains a derivation of maximal order two we can restrict our considerations to second prolongations. The details of higher prolongations can be found in Olver (1986) or Bluman and Kumei (1989).

Consider now a system of sth order PDEs

$$\Delta(x, u, u_1 \ldots, u_s) = 0 \tag{19}$$

where $\Delta = (\Delta^1, \ldots, \Delta^p) = 0$ determines a submanifold in $(x, u, u_1, \ldots, u_s)$-space. If this manifold is invariant under the action of the sth prolongation G_s of G, we say that the system of differential equations (19) is invariant under the group G. Let $\boldsymbol{v}$ be the vector field of G. Then the system (19) is invariant under the group G if and only if

$$\mathbf{pr}^{(s)}\boldsymbol{v}(\Delta)|_{\Delta=0} = 0 \tag{20}$$

where $\mathbf{pr}^{(s)}\boldsymbol{v}$ is the sth prolongation of the vector field $\boldsymbol{v}$. Condition (20) is the defining equation for the group admitted by the system (19) and contains an explicit algorithm, the so-called Lie algorithm (Olver 1986), to determine the infinitesimals ξ and η.

Let us apply the considerations given above to our problem

$$\Delta = u_{xt} - Bx^{\gamma}u - Ax^{\gamma+1}u_x = 0. \tag{21}$$

In order to determine the point transformations (12) under which (21) is invariant one has to consider the vector field $\boldsymbol{v}$ of the independent and dependent variables (x, t, u) in the form

$$\boldsymbol{v} = \xi(x, t, u)\frac{\partial}{\partial x} + \tau(x, t, u)\frac{\partial}{\partial t} + \phi(x, t, u)\frac{\partial}{\partial u}. \tag{22}$$

Applying the second prolongation of $\boldsymbol{v}$ to (21) under the restriction $\Delta = 0$ we get a system of overdetermined equations from which we can determine ξ, τ, and η. The relation $\mathbf{pr}^{(2)}\boldsymbol{v}(\Delta)|_{\Delta=0} = 0$ can be cast into

$$\eta^{xt} - B\gamma x^{\gamma-1}\xi u - Bx^{\gamma}\eta - A(\gamma+1)x^{\gamma}\xi u_x - Ax^{\gamma+1}\eta^x = 0 \tag{23}$$

where η^{xt} and η^x are the second and first prolongation elements needed to determine the infinitesimals ξ and η. We changed notation of (17) and (18) by raising the indices of the independent variables to distinguish numeration from differentiation. Solving (23) for ξ, τ and η one obtains the vector field (22) in an explicit form. In our case the vector field (22) can be derived from a linear combination of the vectors

$$\begin{aligned}
\boldsymbol{v}_1 &= \frac{1}{x^{\gamma}}\frac{\partial}{\partial x} + A(\gamma+1)tu\frac{\partial}{\partial u} \\
\boldsymbol{v}_2 &= -\frac{x}{\gamma+1}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} \\
\boldsymbol{v}_3 &= u\frac{\partial}{\partial u} \\
\boldsymbol{v}_4 &= \frac{\partial}{\partial t} \\
\boldsymbol{v}_5 &= \phi_2(x, t)\frac{\partial}{\partial u}.
\end{aligned} \tag{24}$$

These five linear independent vector fields determine the symmetries under which (21) is invariant. The vector fields $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ contain the scaling properties of (21), $\boldsymbol{v}_4$ corresponds to a translation in time and $\boldsymbol{v}_5$ is characteristic for the linear nature of (21). $\phi_2(x, t)$ of $\boldsymbol{v}_5$ is an arbitrary solution of (21) and demonstrates the superposition principle of linear equations. Thus the corresponding Lie-algebra of infinitesimal symmetries of (21) is spanned by the four vector fields $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_4$ and the infinite-dimensional subalgebra $\boldsymbol{v}_5 = \phi_2(x, t)\partial/\partial u$. The commutation relations of these vector

fields are given in table 1. The entry in row i and column j is $[v_i, v_j]$, respectively. From table 1 it can be detected that the antisymmetry of the algebra and the Jacobi identity are given.

In general to each subgroup H of a symmetry group G of a system of differential equations there will correspond a family of group-invariant solutions. Since there are almost always an infinite number of such subgroups, it is usually not feasible to list all possible group-invariant solutions. Nevertheless we need a systematic procedure of classifying these solutions which Olver (1986) calls an 'optimal system' from which every other solution can be derived. Since elements $g \in G$ not in the subgroup H will transform an H-invariant solution to some other group-invariant solution, only these solutions not so related have to be listed in an optimal system.

The key point of these considerations is to find such symmetry solutions which cannot be transformed to each other by a symmetry transformation. This can be accomplished by introducing the adjoint representation of the Lie algebra. An adjoint representation can be derived by Lie series with

$$\text{Ad } g(w) = \text{Ad}(\exp(\varepsilon v))w = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} (\text{ad } v)^n w \tag{25}$$

with ad $v(w) = [w, v]$ (cf Olver 1986). To demonstrate the determination of the adjoint algebra from the Lie algebra we give the mapping of v_2 by the element connected with v_1

$$\begin{aligned} \text{Ad } e^{\varepsilon v_1} v_2 &= v_2 - \varepsilon[v_1, v_2] + \tfrac{1}{2}\varepsilon^2[v_1, [v_1, v_2]] + \dots \\ &= v_2 + \varepsilon v_1 \end{aligned} \tag{26}$$

where $[v_1, v_2] = -v_1$ can be directly taken from table 1. The adjoint representation of the Lie algebra is summarized in table 2.

Because a linear combination of the five vector fields determines the general symmetry of (21) we can use a combination of the vector fields to classify the types of solution. By using the adjoint algebra we are able to distinguish five different types

Table 1. Lie algebra of the kinetic equation.

$[,]$	v_1	v_2	v_3	v_4
v_1	0	$-v_1$	0	$-\alpha(\gamma+1)v_3$
v_2	v_1	0	0	$-v_4$
v_3	0	0	0	0
v_4	$\alpha(\gamma+1)v_3$	v_4	0	0

Table 2. Adjoint representation of the Lie-algebra.

Ad	v_1	v_2	v_3	v_4
v_1	v_1	$v_2 + \varepsilon v_1$	v_3	$v_4 + \varepsilon\alpha(\gamma+1)v_3$
v_2	$v_1 e^{-\varepsilon}$	v_2	v_3	$v_4 e^{\varepsilon}$
v_3	v_1	v_2	v_3	v_4
v_4	$v_1 - \varepsilon\alpha(\gamma+1)v_3$	$v_2 - \varepsilon v_4$	v_3	v_4

of solutions. Using the adjoint representation the basic fields of an optimal system are given by

$$v^{i} = v_1 + v_2 + \kappa v_4$$

$$v^{ii} = v_1 + v_4$$

$$v^{iii} = v_1$$

$$v^{iv} = v_2 + v_3$$

$$v^{v} = v_4.$$

In the following, we demonstrate that these combinations of symmetries produce essential types of solutions. We mention that one obtains further solutions of (21) by applying finite group transformations to these solutions (Olver 1986).

4. Group invariant solutions

In order to obtain the group invariant solutions (similarity solutions) let us first consider the combination of v_2 and v_3 by

$$w = \lambda v_2 + v_3. \tag{27}$$

We introduce here the parameter λ to demonstrate that not only v^{iv} will give a specific similarity solution but also a linear combination of v_2 and v_3 with arbitrary coefficients. We choose first this simple combination of vector fields for clarity of representation. As we will see, any other combination of vector fields containing v_2 will lead to the same solution type. The corresponding finite transformation to (27) reads

$$\tilde{x} = x\, e^{-\lambda\varepsilon/(\gamma+1)} \qquad \tilde{t} = t\, e^{\lambda\varepsilon} \qquad \tilde{u} = u\, e^{\varepsilon} \tag{28}$$

where ε is the group parameter. The group invariant combinations of these relations are

$$\tilde{t}\tilde{x}^{\gamma+1} = tx^{\gamma+1} \qquad \tilde{t}^{1/\lambda}/\tilde{u} = t^{1/\lambda}/u \tag{29}$$

which suggest the similarity variable ζ and the similarity solution $v(\zeta)$ to be

$$\zeta = tx^{\gamma+1} \qquad u(x, t) = t^{1/\lambda} v(\zeta). \tag{30}$$

Substitution of the similarity solution into (21) results in

$$\zeta v_{\zeta\zeta} + \left\{\frac{1}{\lambda} + 1 - A\zeta\right\} v_{\zeta} - \frac{B}{\gamma+1}\, v = 0. \tag{31}$$

To reduce (31) to the standard form of Kummers's equation we rescale ζ by $z = A\zeta$

$$zv'' + \left\{\frac{1}{\lambda} + 1 - z\right\} v' - \frac{B}{A(\gamma+1)}\, v = 0. \tag{32}$$

The complete solution of Kummer's equation is given by

$$v(z) = C_1 M(a, b, z) + C_2 U(a, b, z) \tag{33}$$

where $a = B/A(\gamma+1))$, and $b = 1/\lambda + 1$, C_1 and C_2 are arbitrary constants, and $b \neq n$. For non-integer values of b this solution can be given by Pochhammer's function, also

called the confluent hypergeometric function, which consists of a convergent series for all z. An alternative representation of the solution (33) turns out to be

$$v(z) = C_1 M(a, b, z) + C_2 z^{1-b} M(a-b+1, 2-b, z) \tag{34}$$

where $M(a, b, z)$ is given by the series

$$M(a, b, z) = {}_1F_1(a, b, z) = 1 + \sum_{k=1}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!} \tag{35}$$

and $(a)_k$, $(b)_k$ are Pochhammer's symbols defined by

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} \qquad (b)_k = \frac{\Gamma(b+k)}{\Gamma(b)}. \tag{36}$$

If we consider b to be a natural number then Kummer's equation is an eigenvalue problem in which $v(z)$ is limited for $z \to 0$ and might not increase stronger than a power of z for $z \to \infty$. The eigenvalues Λ of

$$zv'' + v'\{m - z\} + \Lambda v = 0 \tag{37}$$

are then given by $\Lambda = n - m + 1$ with $n = m-1, m, \ldots$. The corresponding eigenfunctions are generalized Laguerre polynomials $L_n^{(m-1)}(z)$. The eigenvalues for $m = b = 2$ and $a = 1 + 2/(d(\gamma+1))$ determine the product $d(\gamma+1) = -2/n$ and the generalized Laguerre polynomial is $L_n^{(1)}(z)$. We mention that a restriction of the vector field given in the adjoint representation $\boldsymbol{v}^{iv} = \boldsymbol{v}_2 + \boldsymbol{v}_3$ ($\lambda = 1$) will lead to the solutions discussed by McGrady and Ziff (1987) and Corngold and Williams (1989). Kummer's type of solution is also obtained for the vector fields $\boldsymbol{v}_2$ including only scale invariance with respect to x and t. The corresponding similarity representation is given by

$$\zeta = tx^{\gamma+1} \qquad u(x, t) = v(\zeta) \tag{38}$$

and the reduced equation of (21) reads

$$\zeta v_{\zeta\zeta} + (1 - A\zeta) v_\zeta - \frac{B}{\gamma+1} v = 0. \tag{39}$$

A scaling of ζ with $z = A\zeta$ will give the standard form of Kummer's equation. The similarity representation of the combination $\boldsymbol{v}^i = \boldsymbol{v}_1 + \boldsymbol{v}_2 + \kappa \boldsymbol{v}_4$ yields

$$\begin{aligned} &\zeta = \{(\gamma+1) - x^{\gamma+1}\}(t+\kappa) \\ &u(x, t) = \frac{1}{(t+\kappa)^{\kappa A(\gamma+1)}} \exp[\kappa A(\gamma+1)(t+\kappa)] v(\zeta). \end{aligned} \tag{40}$$

This solution inserted in (21) gives also Kummer's equation. A subgroup of $\boldsymbol{v}^i$ with $\kappa = 0$ is obtained by combining $\boldsymbol{v}_1 + \boldsymbol{v}_2$ which also gives Kummer's equation after reduction of (21). The corresponding similarity representation is given in table 3.

Another type of similarity reduction can be obtained if we examine the linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_4$ by

$$\boldsymbol{w} = \boldsymbol{v}_1 + \lambda \boldsymbol{v}_4. \tag{41}$$

Here we introduce λ in a similar way as above. The finite transformation for this combination can be written as

$$\begin{aligned} &\tilde{x}^{\gamma+1} = (\gamma+1)\varepsilon + x^{\gamma+1} \qquad \tilde{t} = \lambda\varepsilon + t \\ &\tilde{u} = u \exp[A(\gamma+1)(\tfrac{1}{2}\lambda\varepsilon^2 + t\varepsilon)]. \end{aligned} \tag{42}$$

Table 3. Summary of similarity solutions for the fragmentation equation (12), ($\mu = \gamma + 1$).

Case	Group	ζ	$u(x,t)$	Solution for v
(1)	v_2	tx^μ	$v(\zeta)$	Kummer's function ${}_1F_1(a,b,z)$
	$\lambda v_2 + v_3$	tx^μ	$t^{1/\lambda} v(\zeta)$	
	$v_1 + v_2$	$t\{\mu - x^\mu\}$	$e^{A\mu t} v(\zeta)$	
	$v_1 + v_2 + \kappa v_4$	$(t+\kappa)\{\mu - x^\mu\}$	$\dfrac{1}{(t+\kappa)^{\kappa A\mu}} e^{A\mu(t+\kappa)} v(\zeta)$	
(2)	$v_1 + \lambda v_4$	$x^\mu - ct$	$e^{\frac{1}{2}Ac t^2} v(\zeta)$	Fox function
(3)	v_1	t	$e^{Atx^\mu} v(\zeta)$	$Dt^{2/\mu}$
(4)	v_4	x	$v(\zeta)$	$x^{-B/A}$

If we use the special solution $u(x) = 1/x^{B/A}$ we obtain from (42) another solution $u(x, t)$ which reads

$$u(x,t) = \frac{C}{(x^{\gamma+1} - (\gamma+1)\varepsilon)^{B/A(\gamma+1)}} \exp[A(\gamma+1)(-\tfrac{1}{2}\lambda\varepsilon^2 + t\varepsilon)] \tag{43}$$

where C is an arbitrary constant. The general reduction of this subgroup can be obtained by the similarity representation

$$\zeta = x^{\gamma+1} - ct \qquad u(x,t) = \exp(\tfrac{1}{2}Act^2) v(\zeta) \tag{44}$$

where c is given by $c = (\gamma+1)/\lambda$. The corresponding ordinary differential equation is

$$v_{\zeta\zeta} - \frac{d}{c}\zeta v_\zeta - \frac{d(\gamma+1)+2}{c(\gamma+1)} v = 0. \tag{45}$$

where we have replaced A and B by the definitions used above ($A = -d$ and $B = -\{d(\gamma+1)+2\}$). A scaling of the similarity variable ζ by $z = \sqrt{(d/c)}\zeta$ will again transform (45) to a standard equation (Kamke 1977)

$$v'' - zv' - av = 0 \tag{46}$$

where primes denote differentiation with respect to z, and a is given by $a = (d(\gamma+1)+2)/(d(\gamma+1))$. The general solution of this ODE can be given by a special type of Fox function (Braaksma 1964)

$$\begin{aligned} v(z) = C_1 H^{12}_{23}\left(-2z^2 \middle| \begin{matrix}(0,1) & (1-\frac{1}{2}a, 1) \\ (0,1) & (0,2) \quad (1-\frac{1}{2}a, 0)\end{matrix}\right) \\ + C_2 z H^{12}_{23}\left(-2z^2 \middle| \begin{matrix}(0,1) & (\frac{1}{2}(1-a), 1) \\ (0,1) & (-1,2) \quad (\frac{1}{2}(1-a), 0)\end{matrix}\right) \end{aligned} \tag{47}$$

or equivalently by Maitland's generalized hypergeometric functions, which are also called Wright functions:

$$v(z) = C_1 {}_2\Psi_3\left(\begin{matrix}(1,1) & (\frac{1}{2}a, 1) \\ (1,2) & (\frac{1}{2}a, 0)\end{matrix}; 2z^2\right) + C_2 z\, {}_2\Psi_3\left(\begin{matrix}(1,1) & (\frac{1}{2}(a+1), 1) \\ (2,2) & (\frac{1}{2}(a+1), 0)\end{matrix}; 2z^2\right). \tag{48}$$

The Wright function (Mathai and Saxena 1978) are defined as

$${}_p\Psi_q\left(\begin{matrix}(a_1,\alpha_1) & \dots & (a_p,\alpha_p) \\ (b_1,\beta_1) & \dots & (b_q,\beta_q)\end{matrix}; z\right) = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(a_j + \alpha_j k)}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j k)} \frac{(-z)^k}{k!}. \tag{49}$$

If we choose the parameters to be $d/c=2$ and $-(d(\gamma+1)+2)/(c(\gamma+1))$, equation (45) can be solved by Hermite polynomials. The given restrictions for the parameters in this case can be summarized in $d\lambda=2(\gamma+1)$ and $\lambda=-(n+1)(\gamma+1)^2$. If we transform $v(z)$ on the other hand by $v(z)=u(z)\exp(z^2/2)$ we get

$$u''+zu'+(1-a)u=0 \tag{50}$$

assuming that $(1-a)=n+1$ we can derive solutions $u(z)$ in the form

$$u(z)=\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left\{\exp(-z^2/2)\left(C_1+C_2\int\exp(z^2/2)\,\mathrm{d}z\right)\right\}. \tag{51}$$

A further similarity solution is generated by the vector field v_1. The similarity variable is just the time variable ($\zeta=t$). To the symmetry v_1 corresponds the similarity solution $u(x,t)=\exp(Atx^{\gamma+1})v(\zeta)$. Substitution of this similarity solution into (21) gives the reduction

$$\zeta v_\zeta=\left(\frac{B}{A(\gamma+1)}-1\right)v. \tag{52}$$

This ODE can be integrated by a separation of variables to $v(\zeta)=D\zeta^{B/[A(\gamma+1)]}$.

The last vector field of our optimal system which remains to be discussed is $v^\nu=v_4$. Since we have here a translational symmetry in time it is easy to get the similarity representation by $\zeta=x$ and $u(x,t)=v(\zeta)$. The reduced equation for this class follows straightforward from (21) to be

$$A\zeta v_\zeta+Bv=0. \tag{53}$$

An integration gives $v(\zeta)=C\zeta^{-B/A}$ with C as a constant of integration.

A summary of the different similarity solutions ordered by the type of ODE is given in table 3. In table 3 we also give the similarity representation of the different symmetry groups following from the optimal system. We see that a classification by ODEs is compatible with the optimal system.

By performing the similarity analysis we are able to classify four types of solutions for equation (11). Three subgroups reduce to Kummer's equation and three others to Weber's ODE (Fox function), a scaling solution in the time variable, and spatial scaling, respectively. If we have the solutions on hand we can ask for the asymptotic behaviour $t\to\infty$ of the similarity solutions. For $t\to\infty$ and $tx^{\gamma+1}=\omega=$ constant we obtain for case (1) of table 3

$$\lim_{t\to\infty}n(x,t)=\begin{cases}x^\nu v(\omega) & \text{for } v_2 \text{ and model (ii), (iii)}\\ v(\omega) & \text{for } v_2 \text{ and model (i)}\end{cases} \tag{54}$$

$$\lim_{t\to\infty}n(x,t)=\begin{cases}0 & \text{for } \lambda v_2+v_3 \text{ and } \lambda<0\\ \infty & \text{for } \lambda v_2+v_3 \text{ and } \lambda>0\end{cases} \tag{55}$$

$$\lim_{t\to\infty}n(x,t)=0 \qquad \text{for } v_1+v_2+\kappa v_4 \text{ for all models.} \tag{56}$$

For case (2) the solution $n(x,t)$ tends to 0 for $t\to\infty$. In case (3) we have

$$\lim_{t\to\infty}n(x,t)=D\,\mathrm{e}^{A\omega}\lim_{t\to\infty}t^{2/(\gamma+1)}$$
$$=\begin{cases}\infty & \text{for model (i)}\\ \left.\begin{cases}0 & \text{for } -<\nu<-1\\ \infty & \text{for } \nu>-1 \text{ or } \nu<-2\end{cases}\right\} & \text{for model (ii), (iii).}\end{cases} \tag{57}$$

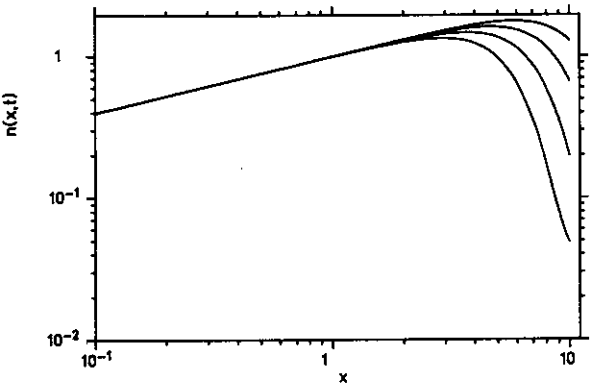

Figure 1. Scaling behaviour for the similarity solution of Kummer type in case of model (ii). The model parameters are $\nu = 4/10$, $\beta = 2.0$, $\lambda = 1.0$ and $x \in [0, 10]$. The different curves correspond, from top to bottom, to times $t = 1 \times 10^{-3}$, 2×10^{-3}, 4×10^{-3}, 8×10^{-3}, respectively.

We see that only for model (ii) and (iii) a scaling behaviour like $n(\lambda x, t \to \infty)|_{tx^{\nu+1}=\omega} = \lambda^{\nu} x^{\nu}$ is present. In figure 1 we have plotted a similarity solution of Kummer type for model (ii) under the restriction $C_1 = 1$ and $C_2 = 0$. We plotted the similarity solution for four fixed times t. Even for a fixed t we can detect a scaling regime which gives $n(x, t) \sim x^{\nu}$ for $x \to 0$.

5. Conclusions

We demonstrated that by applying Lie's similarity method to the linear PDE (8) resulting from the integro-differential equation (5) a great variety of solutions is obtained. Some of these solutions for the initial value problem $n(x, t = 0) = \delta(x - L)$ were discussed by McGrady and Ziff (1987). These authors made a special ansatz for the solution to obtain Kummers solution. Our procedure delivers these special type of solution and several other for fragmentation processes of different type. We find that our solutions show a scaling behaviour for small spatial values if we consider breakup kernel (ii) and (iii). For this spatial regime we demonstrated that $n(x, t)$ behaves asymptotically as x^{ν} for $x \to 0$.

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